JOURNAL OF FUNCTIONAL ANALYSIS 10, 19–32 (1972)

Classes of C^∞ Vectors and Essential Self-Adjointness*

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Communicated by the Editors

Received February 6, 1971

Criteria for self-adjointness are proved using methods related to the moment problem. For the special case of semibounded operators, we obtain a stronger result than the analytic or quasianalytic vector criteria. The results are applied to certain approximate quantum field theory Hamiltonians.

I. INTRODUCTION

Nelson's theorem [1] on analytic vectors has proven useful in finding domains of essential self-adjointness for operators which arise in quantum field theory [2, 3]. Nussbaum [4] has found larger classes of vectors for which similar theorems hold. In Section II, we give simple proofs of these theorems and establish a stronger result concerning the "Stieltjes vectors" of semibounded operators. In Section III, we show how the latter can be used to prove that certain approximate quantum field theory Hamiltonians are essentially self-adjoint. In Section IV, we indicate how the theorems in Section II lead to a solution of linear equations by the method of moments or by Padé approximants.

* Supported in part by the National Research Council of Canada.

II. Stieltjes Vectors

In the following, A will denote a closed symmetric (in general, unbounded) operator with domain $\mathscr{D}(A)$ and range $\mathscr{R}(A)$ in a Hilbert space $\mathscr{H}$. Let $C^\infty(A) = \bigcap_{n=1}^\infty \mathscr{D}(A^n)$. Let $f \in C^\infty(A)$ and let $\mathscr{L}_f$ be the vector space spanned by the vectors $\{A^k f \mid k = 0, 1,...\}$. Let $\mathscr{H}_f = \bar{\mathscr{L}}_f$, the Hilbert space formed by completion of $\mathscr{L}_f$. The restriction of A to $\mathscr{L}_f$ has a unique minimal closed extension $B = (A \mid \mathscr{L}_f)^-$. B is a restriction of the symmetric operator A (we write $B \subseteq A$) and thus it is a closed, symmetric operator with domain $\mathscr{D}(B)$ dense in $\mathscr{H}_f$. Note that if z is a complex number, then $\mathscr{R}(z - B) \subseteq \mathscr{R}(z - A)$.

We use a common notation for the norm and inner product in $\mathscr{H}$: $\langle f \mid \alpha g\rangle = \alpha\langle f \mid g\rangle$; $\langle f \mid f\rangle = \| f \|^2$.

Let us define the following classes of vectors in $C^\infty(A)$:

(i) Vectors of uniqueness:

$$\mathscr{U} = \{f \mid B = (A \mid \mathscr{L}_f)^- \text{ is a self-adjoint operator in } \mathscr{H}_f\}.$$

(ii) Stieltjes vectors: $\mathscr{S} = \{f \mid \sum_{n=1}^\infty \| A^n f \|^{-1/2n} = \infty\}$.

(iii) Quasi-analytic vectors: $\mathscr{Q} = \{f \mid \sum_{n=1}^\infty \| A^n f \|^{-1/n} = \infty\}$.

(iv) Analytic vectors:

$$\mathscr{A} = \left\{f \,\middle|\, \sum_{n=1}^\infty \| A^n f \| s^n/n! < \infty \text{ for some } s > 0\right\}.$$

It is not hard to see that $\mathscr{A} \subseteq \mathscr{Q} \subseteq \mathscr{S}$. Roughly speaking $\mathscr{A}$ and $\mathscr{Q}$ contain vectors f such that $\| A^n f \|$ increases less rapidly than $n!$ or n^n for large n. $\mathscr{S}$ contains vectors f such that $\| A^n f \|$ increases like n^{2n}. $\mathscr{A}$ is clearly a linear space, but it is not clear whether this is true of $\mathscr{Q}$ or $\mathscr{S}$. Nussbaum [4] states that $\mathscr{U}$ is not a linear space.

Throughout this section we will use the following characterization of self-adjoint operators [5]: If A is a closed, symmetric operator in a Hilbert space $\mathscr{H}$, then, in order that A be self-adjoint, it is necessary that $\mathscr{R}(z - A) = \mathscr{H}$ for all nonreal z and sufficient that $\mathscr{R}(z - A)$ be dense in $\mathscr{H}$ for some z in each half plane, Im $z > 0$, Im $z < 0$.

We may thus write an equivalent definition of vectors of uniqueness:

$$\mathscr{U} = \{f \in C^\infty(A) \mid \mathscr{R}(z - B) = \mathscr{H}_f \text{ for all nonreal } z\}.$$

Lemma 1. *A closed, symmetric operator A is self-adjoint if it has a dense set $\mathscr{U}$ of vectors of uniqueness.*

Proof. By the characterization above it is enough to show that if $\operatorname{Im} z \neq 0$, then $\mathscr{U} \subseteq \mathscr{R}(z - A)$. If $f \in \mathscr{U}$, then $\mathscr{R}(z - B) = \mathscr{H}_f$, so that $f \in \mathscr{R}(z - B)$. But we have noted that $\mathscr{R}(z - B) \subseteq \mathscr{R}(z - A)$, so that $f \in \mathscr{R}(z - A)$.

In the following, we will assume that $\mathscr{L}_f$ is not of finite dimension. The case of finite dimension will be treated separately in the proofs of Lemmas 3 and 6.

The Gramm–Schmidt orthogonalization procedure applied to the sequence $\{A^n f \mid n = 0, 1,...\}$ yields a three-term recursion relation for the orthogonal vectors $\{f_n \mid n = 0, 1,...\}$:

$$f_0 = f,$$

$$f_1 = (A - a_0) f_0,$$

$$f_{i+1} = (A - a_i) f_i - \frac{w_i^2}{w_{i-1}^2} f_{i-1} \qquad (i = 1, 2,...),$$

where $w_i = \| f_i \|$ and $a_i = \langle f_i \mid A f_i \rangle / w_i^2$ is real for $i = 0, 1,...$.

Note that $w_n = \| f_n \| \leqslant \| A^n f \|$ for $n = 0, 1,...$. It is more convenient to use the complete orthonormal basis $\{e_n \mid n = 0, 1,...\}$ in $\mathscr{H}_f$ defined by:

$$\begin{aligned} e_i &= f_i / w_i, && i = 0, 1,..., \\ A e_i &= b_{i-1} e_{i-1} + a_i e_i + b_i e_{i+1}, && i = 0, 1,..., \end{aligned} \tag{1}$$

where $b_{-1} = 0$ and $b_i = w_{i+1}/w_i > 0$ for $i = 0, 1,...$. Let $p_0 = 1$ and let $\{p_n(\lambda) \mid n = 1, 2,...\}$ be defined by the relations

$$\lambda p_n = b_{n-1} p_{n-1} + a_n p_n + b_n p_{n+1}, \qquad n = 0, 1,... . \tag{2}$$

Note that $p_n(\bar\lambda) = \overline{p_n(\lambda)}$.

LEMMA 2. *If* $\sum_{n=1}^\infty | p_n(\lambda)|^2 = \infty$ *for some nonreal* λ, *then* f *is a vector of uniqueness for* A.

Proof. Suppose there is a nonzero vector $g \in \mathscr{H}_f$ orthogonal to $\mathscr{R}(\bar\lambda - B)$. Then,

$$\langle (\bar\lambda - B) e_n \mid g \rangle = 0, \qquad n = 0, 1,... .$$

Using (1), this gives

$$\begin{aligned} \lambda \langle e_n \mid g \rangle &= \langle B e_n \mid g \rangle \\ &= b_{n-1} \langle e_{n-1} \mid g \rangle + a_n \langle e_n \mid g \rangle + b_n \langle e_{n+1} \mid g \rangle, \qquad n = 0, 1,... . \end{aligned}$$

Taking $p_n = \langle e_n \mid g \rangle$, we see that the $\{p_n\}$ satisfy (2) and $g = \sum_{n=0}^{\infty} p_n e_n$ (we may assume that g is normalized so that $p_0 = \langle e_0 \mid g \rangle = 1$, since if $p_0 = 0$, then (2) implies that $p_n = 0$ for $n = 1, 2, \ldots$). If $\sum_{n=0}^{\infty} \mid p_n(\lambda)|^2 = \infty$, no such vector g exists, and hence $\mathscr{R}(\bar{\lambda} - B)$ is dense in $\mathscr{H}_f$. Since $| p_n(\bar{\lambda})| = | p_n(\lambda)|$, we find similarly that $\mathscr{R}(\lambda - B)$ is dense in $\mathscr{H}_f$, so that B is self-adjoint in $\mathscr{H}_f$ and $f \in \mathscr{U}$.

From (2) we get

$$\begin{aligned}(\lambda - \mu)\, p_n(\lambda)\, p_n(\mu) &= b_n[p_{n+1}(\lambda)\, p_n(\mu) - p_{n+1}(\mu)\, p_n(\lambda)] \\ &\quad - b_{n-1}[p_n(\lambda)\, p_{n-1}(\mu) - p_n(\mu)\, p_{n-1}(\lambda)].\end{aligned}$$

Remembering that $b_{-1} = 0$, it follows that

$$(\lambda - \mu) \sum_{k=0}^{n} p_k(\lambda)\, p_k(\mu) = b_n[p_{n+1}(\lambda)\, p_n(\mu) - p_{n+1}(\mu)\, p_n(\lambda)]. \tag{3}$$

Taking $\mu = \bar{\lambda}$ and $p_k(\bar{\lambda}) = \bar{p}_k$, we have

$$(\lambda - \bar{\lambda}) \sum_{k=0}^{n} | p_k |^2 = b_n[p_{n+1}\bar{p}_n - \bar{p}_{n+1} p_n]. \tag{4}$$

In the next Lemma we use a form of Carleman's inequality. We give a short proof due to Polya [6].

Let $\{u_k \mid k = 1, 2, \ldots\}$ be a sequence of nonnegative real numbers, not all of which are zero. Define numbers $\{c_k \mid k = 1, 2, \ldots\}$ by the equations $c_1 c_2 \cdots c_n = (n + 1)^n$ so that

$$c_n = (n + 1)^n / n^{n-1} < ne.$$

Using the inequality between the arithmetic and geometric means, we get Carleman's inequality

$$\begin{aligned}\sum_{n=1}^{\infty} (u_1 u_2 \cdots u_n)^{1/n} &= \sum_{n=1}^{\infty} \frac{(u_1 c_1 u_2 c_2 \cdots u_n c_n)^{1/n}}{n + 1} \\ &\leqslant \sum_{n=1}^{\infty} \frac{u_1 c_1 + \cdots + u_n c_n}{n(n + 1)} = \sum_{k=1}^{\infty} u_k c_k \sum_{n=k}^{\infty} \frac{1}{n(n + 1)} \\ &= \sum_{k=1}^{\infty} u_k c_k \sum_{n=k}^{\infty} \left(\frac{1}{n} - \frac{1}{n + 1}\right) = \sum_{k=1}^{\infty} u_k c_k \frac{1}{k} \\ &< e \sum_{k=1}^{\infty} u_k .\end{aligned}$$

LEMMA 3. *If f is a quasianalytic vector for A, then f is a vector of uniqueness for A.*

Proof. We show that if $\sum_{n=1}^{\infty} \| A^n f \|^{-1/n} = \infty$, then the series $\sum_{n=0}^{\infty} | p_k(\lambda)|^2$ diverges for all nonreal λ.

Since $\sum_{k=0}^{n} | p_k(\lambda)|^2 \geqslant | p_0 |^2 = 1$ and $b_n > 0$ for $n = 0, 1, \dots$, we have, from Eq. (4),

$$\frac{1}{b_n} \leqslant \frac{\bar{p}_n p_{n+1} - p_n \bar{p}_{n+1}}{\lambda - \bar{\lambda}}, \qquad n = 0, 1, \dots .$$

Using the Schwarz inequality for sequences:

$$\sum_{n=0}^{\infty} \frac{1}{b_n} \leqslant \frac{1}{\lambda - \bar{\lambda}} \sum_{n=0}^{\infty} (\bar{p}_n p_{n+1} - p_n \bar{p}_{n+1}) \leqslant \frac{1}{|\operatorname{Im} \lambda|} \sum_{n=0}^{\infty} | p_n |^2.$$

The relations $b_n = w_{n+1}/w_n$ and $w_n \leqslant \| A^n f \|$ together with Carleman's inequality (with $u_n = w_n/w_{n+1}$) give us the desired result:

$$\sum_{n=1}^{\infty} \| A^n f \|^{-1/n} \leqslant \sum_{n=1}^{\infty} w_n^{-1/n} < e \sum_{n=1}^{\infty} \frac{w_n}{w_{n+1}} = e \sum_{n=1}^{\infty} \frac{1}{b_n}$$

$$\leqslant (e| \operatorname{Im} \lambda |) \sum_{n=0}^{\infty} | p_k(\lambda)|^2.$$

In the case that $\mathscr{L}_f$ is of finite dimension, B is a self-adjoint, bounded operator in $\mathscr{H}_f$ with bound $\| B \|$. Since $\| A^n f \| = \| B^n f \| \leqslant \| B \|^n \| f \|$, we conclude that f is both a quasianalytic vector and a vector of uniqueness.

We can now state the result:

THEOREM 1. *Let A be a closed, symmetric operator in a Hilbert space $\mathscr{H}$. Then the following statements are equivalent:*

(i) *A is self-adjoint.*

(ii) *A has a dense set $\mathscr{U}$ of vectors of uniqueness.*

(iii) *A has a dense set $\mathscr{Q}$ of quasi-analytic vectors.*

(iv) *A has a dense set $\mathscr{A}$ of analytic vectors.*

Proof. (i) implies that A has a spectral measure $E(\cdot)$. If Φ is a bounded Borel set, then any vector f in the range of $E(\Phi)$ is an analytic vector for A. (If Φ is contained in the interval $[-b, b]$ and f is in the range of $E(\Phi)$, then $\| A^n f \| \leqslant b^n \| f \|$.) The set of such vectors is dense in $\mathscr{H}$. Since $\mathscr{A} \subseteq \mathscr{Q} \subseteq \mathscr{U}$ (Lemma 3), $\mathscr{Q}$ and $\mathscr{U}$ are also dense in $\mathscr{H}$. Conversely, if $\mathscr{A}$, $\mathscr{Q}$ or $\mathscr{U}$ is dense, then $\mathscr{U}$ is dense, so (i) holds by Lemma 1.

The relation between (i) and (iv) is Nelson's theorem [1]. The equivalence of the other two statements was proven by Nussbaum [4] using an equivalent definition of vectors of uniqueness ($B = (A \mid \mathscr{L}_f)^-$ is self-adjoint in $\mathscr{H}_f$ if and only if the sequence $\{\langle f \mid A^n f\rangle \mid n = 0, 1,...\}$ is a determinate Hamburger moment sequence [7]).

A symmetric operator A is said to be semibounded from above (below) if there is a real constant M such that $\langle f \mid Af\rangle < M\langle f \mid f\rangle$ ($> M\langle f \mid f\rangle$) for all f in $\mathscr{D}(A)$. A is said to be semibounded if it is semibounded from above or from below. We show that if A is a semibounded operator, then a Stieltjes vector for A is a vector of uniqueness for A. We will need some facts about the $\{p_n(\lambda)\}$ which will follow from the properties of certain approximate operators.

Let $\mathscr{H}_n$ be the Hilbert space of dimension n spanned by the vectors $\{A^k f \mid k = 0, 1,..., n-1\}$. Let E_n be the orthogonal projection from $\mathscr{H}$ onto $\mathscr{H}_n$ and let $B_n = E_n A E_n$. B_n is a symmetric operator which is equal to A when acting on vectors $A^k f$ or e_k for $k \leqslant n-2$. Thus any vector $g \in \mathscr{H}_n$ may be expressed as $g = R(A)\, e_0 = R(B_n)\, e_0$, where $R(\lambda)$ is a polynomial of degree less than or equal to $n - 1$. By comparing Eqs. (1) and (2), we see that $p_n(\lambda)$ is a polynomial of degree n in λ and that $e_n = p_n(A)\, e_0$. We have that $p_n(B_n)\, e_0 = E_n p_n(A)\, e_0 = E_n e_n = 0$, since e_n is orthogonal to $\mathscr{H}_n$. Consequently, if $g = R(B_n)\, e_0$ is any vector in $\mathscr{H}_n$, then $p_n(B_n)\, R(B_n)\, e_0 = R(B_n)\, p_n(B_n)\, e_0 = 0$.

We conclude that $p_n(B_n) = 0$, and hence that $p_n(\lambda)$ is the characteristic polynomial of the operator B_n in $\mathscr{H}_n$. Since B_n is symmetric, $p_n(\lambda)$ must have n real zeros $\{\lambda_k \mid k = 1,..., n\}$ which are equal to the eigenvalues of B_n in $\mathscr{H}_n$.

LEMMA 4. *If A is semibounded from above by M, then $p_n(\lambda) > 0$ when $\lambda > M$.*

Proof. It is easy to see that B_n is also semibounded from above by M so that the eigenvalues (zeros of $p_n(\lambda)$) $\{\lambda_k \mid k = 1,..., n\}$ are all less than M. Since $p_{n+1}(\lambda) = (\lambda - a_n)\, p_n(\lambda)/b_n - p_{n-1}(\lambda)\, b_{n-1}/b_n$ by (2) and $b_n > 0$, the coefficient of λ^n in $p_n(\lambda)$ is positive; so $p_n(\lambda)$ is strictly positive for $\lambda > M$.

LEMMA 5. *For real λ, w we have*

$$| p_n(\lambda)|^2 \leqslant | p_n(\lambda + iw)|^2.$$

Proof. For some $c > 0$, we have $p_n(\lambda) = c \prod_{k=1}^{n} (\lambda - \lambda_k)$. Then,

$$\begin{aligned} |p_n(\lambda + iw)|^2 &= c^2 \left| \prod_{k=1}^{n} (\lambda + iw - \lambda_k) \right|^2 \\ &= c^2 \prod_{k=1}^{n} [(\lambda - \lambda_k)^2 + w^2] \\ &\geqslant c^2 \prod_{k=1}^{n} (\lambda - \lambda_k)^2 \\ &= |p_n(\lambda)|^2. \end{aligned}$$

LEMMA 6. *If A is semibounded from above by M and f is a Stieltjes vector for A, then f is a vector of uniqueness for A.*

Proof. We show that if $\sum_{n=1}^{\infty} \| A^n f \|^{-1/2n} = \infty$, then $\sum_{k=0}^{\infty} | p_k(z)|^2$ diverges for Re $z > M$.

Let $\lambda > \mu > M$ so that, by Lemma 4, $p_n(\lambda) > 0$ and $p_n(\mu) > 0$ for all n. Since $\sum_{k=0}^{n} p_k(\lambda)\, p_k(\mu) \geqslant p_0(\lambda)\, p_0(\mu) = 1$ and $b_n > 0$ we have, from Eq. (3),

$$\frac{1}{b_n p_n(\lambda)\, p_{n+1}(\lambda)} \leqslant \frac{1}{\lambda - \mu} \left[\frac{p_n(\mu)}{p_n(\lambda)} - \frac{p_{n+1}(\mu)}{p_{n+1}(\lambda)} \right].$$

Since $b_n p_n(\lambda)\, p_{n+1}(\lambda) > 0$ and $\lambda - \mu > 0$, the positive terms $p_n(\mu)/p_n(\lambda)$ must decrease monotonically to some limit $c \geqslant 0$ and $\sum_{n=0}^{\infty} [b_n p_n(\lambda)\, p_{n+1}(\lambda)]^{-1} \leqslant (1 - c)/(\lambda - \mu)$. Knowing that this sequence converges, we can use the Schwarz inequality to write

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{b_n^{1/2}} &= \sum_{n=0}^{\infty} \frac{1}{[b_n p_n(\lambda)\, p_{n+1}(\lambda)]^{1/2}} [p_n(\lambda)\, p_{n+1}(\lambda)]^{1/2} \\ &\leqslant \left[\sum_{n=0}^{\infty} \frac{1}{b_n p_n(\lambda)\, p_{n+1}(\lambda)} \right]^{1/2} \left[\sum_{n=0}^{\infty} p_n(\lambda)\, p_{n+1}(\lambda) \right]^{1/2} \\ &\leqslant \left(\frac{1 - c}{\lambda - \mu} \right)^{1/2} \left[\sum_{n=0}^{\infty} p_n(\lambda)\, p_{n+1}(\lambda) \right]^{1/2} \\ &\leqslant \left(\frac{1 - c}{\lambda - \mu} \right)^{1/2} \left[\sum_{n=0}^{\infty} | p_n(\lambda)|^2 \right]^{1/2}. \end{aligned} \tag{5}$$

As in the proof of Lemma 3

$$\sum_{n=1}^{\infty} \| A^n f \|^{-1/2n} \leqslant \sum_{n=1}^{\infty} w_n^{-1/2n} < e \sum_{n=1}^{\infty} \left(\frac{w_n}{w_{n+1}} \right)^{1/2} = e \sum_{n=1}^{\infty} \frac{1}{b_n^{1/2}}. \tag{6}$$

Combining (5) and (6) and using Lemma 5,

$$\sum_{n=1}^{\infty} \| A^n f \|^{-1/2n} \leqslant e \sum_{n=1}^{\infty} \frac{1}{b_n^{1/2}} \leqslant e \left(\frac{1-c}{\lambda - \mu} \right)^{1/2} \left(\sum_{n=0}^{\infty} | p_n(\lambda) |^2 \right)^{1/2}$$

$$\leqslant e \left(\frac{1-c}{\lambda - \mu} \right)^{1/2} \left(\sum_{n=0}^{\infty} | p_n(\lambda + iw) |^2 \right)^{1/2}.$$

Taking w positive or negative, we see that $\sum_{n=0}^{\infty} | p_n(z) |^2$ diverges for any point z in either half-plane for which $\operatorname{Re}(z) = \lambda > M$.

The case when $\mathscr{L}_f$ is of finite dimension is treated as in Lemma 3.

Since the Stieltjes vectors and vectors of uniqueness of A are identical to those of $(-A)$, this Lemma applies also to operators which are bounded from below.

THEOREM 2. *Let A be a closed, symmetric, semi-bounded operator in a Hilbert space $\mathscr{H}$. Then the following statements are equivalent:*

(i) *A is self-adjoint*

(ii) *A has a dense set $\mathscr{S}$ of Stieltjes vectors.*

Proof. If (i) holds, then $\mathfrak{A}$ is dense. Since $\mathfrak{A} \subseteq \mathscr{S}$, $\mathscr{S}$ must also be dense in $\mathscr{H}$. Conversely, if $\mathscr{S}$ is dense then $\mathscr{U}$ is dense by Lemma 6. By Lemma 1, A is self-adjoint.

Note that if a symmetric, semi-bounded (not necessarily closed) operator T has a dense set $\mathscr{S}$ of Stieltjes vectors, then T is essentially self-adjoint.

III. APPLICATIONS

As a simple example we show that the "anharmonic oscillator" Hamiltonian

$$H = H_0 + H_I = \frac{1}{2} \left(- \frac{d^2}{dx^2} + x^2 - 1 \right) + \lambda x^4,$$

where $\lambda > 0$, is essentially self-adjoint on the space $\mathscr{D}$ spanned by the eigenvectors $\{e_m \mid m = 0, 1,...\}$ of H_0. H is clearly semibounded from below.

We write:

$$\left.\begin{aligned} a &= 2^{-1/2}\left(x + \frac{d}{dx}\right), & a^+ &= 2^{-1/2}\left(x - \frac{d}{dx}\right), \\ ae_m &= m^{1/2}e_{m-1}, & a^+e_m &= (m+1)^{1/2}\, e_{m+1}, \\ H_0 e_m &= a^+ae_m = me_m . & & \end{aligned}\right\} \tag{7}$$

Then $H = a^+a + (\lambda/4)(a + a^+)^4$. It is convenient to write $H = \sum_k W_k$ with $[H_0, W_k] = kW_k$. Here $|k| = 4$, 2, or 0 and, for example, $W_4 = (\lambda/4)\, a^+a^+a^+a^+$ and $W_0 = a^+a + (\lambda/4)(a^+aaa^+ + 5 \text{ other terms})$.

Using (7), it is not hard to check that the inequality

$$\| W_k\psi \| < c\,\|(H_0 + 4)^2\psi\|$$

holds for $|k| = 4, 2, 0$, $\psi \in \mathscr{D}$, and $c = (1/8 + 3\lambda/2)$.

H is essentially self-adjoint on $\mathscr{D}$ by the following corollary of Theorem 2. The conditions (i)–(vi) of the corollary are satisfied if we take $N = H_0$, $C = \mathscr{D}$ and $W = H$.

COROLLARY 1. *Let the operators N and W_k, $|k| = 4, 2, 0$ satisfy the conditions:*

(i) *N is a number operator with eigenvalues* 0, 1,... .

(ii) *There is a domain C contained in the linear span of the eigenvectors of N. C is a common invariant domain for N and W_k.*

(iii) $[N, W_k] = kW_k$ *for* $|k| = 4, 2, 0$.

(iv) *There is a constant c such that* $\| W_k\Phi \| \leqslant c\,\|(N + 4)^2\, \Phi\|$ *for* $|k| = 4, 2, 0$ *and for all* $\Phi \in C$.

(v) *$W = \sum_k W_k$ is a semibounded symmetric operator.*

(vi) *C is a dense set.*

or (vi)′ *Conditions* (i)–(iv) *hold for a number of domains C_L with different constants c_L and $\bigcup\{C_L\}$ is a dense set. Then W is essentially self-adjoint on C (or $\bigcup\{C_L\}$).*

Proof. For $i = 1, 2,...$, let $|k_i| = 4, 2, 0$. Using (iii) and (iv) with $\Phi \in C$,

$$\| W_{k_1}\Phi \| \leqslant c\,\|(N + 4)^2\Phi\|$$

$$\begin{aligned} \| W_{k_1}W_{k_2}\Phi \| &\leqslant c\,\|(N + 4)^2\, W_{k_2}\Phi\| \\ &= c\,\| W_{k_2}(N + 4 + k_2)^2\Phi\| \\ &\leqslant c^2\,\|(N + 4)^2\,(N + 4 + k_2)^2\Phi\|. \end{aligned}$$

By induction we get

$$\| W_{k_1} \cdots W_{k_n} \Phi \| \leqslant c^n \|(N+4)^2 (N+4+k_n)^2 \cdots (N+4+k_n+\cdots+k_2)^2 \Phi \|.$$

Noting that any vector $\Phi \in C$ has the form $\Phi = \sum_{j=0}^{M} a_j \Phi_j$ with $N\Phi_m = m\Phi_m$ and that $| k_i | \leqslant 4$, we have

$$\begin{aligned} \| W_{k_1} \cdots W_{k_n} \Phi \| &\leqslant c^n (M+4)^2 \cdots (M+4+k_n+\cdots+k_2)^2 \| \Phi \| \\ &\leqslant c^n (M+4)^2 \cdots (M+4n)^2 \| \Phi \| \\ &\leqslant c^n (M+4n)^{2n} \| \Phi \|. \end{aligned}$$

Since W^n is a sum of 5^n terms of the form $W_{k_1} \cdots W_{k_n}$, we get:

$$\| W^n \Phi \| \leqslant 5^n c^n (M+4n)^{2n} \| \Phi \|.$$

Thus any vector in C (or $\bigcup\{C_L\}$) is a Stieltjes vector for W. Since C (or $\bigcup\{C_L\}$) is dense and W is semibounded, W is essentially self-adjoint by Theorem 2.

This example suggests that we try to apply the method to the Hamiltonian of a quantum field theory with a quartic interaction. Consider the Hamiltonian $H = H_0 + H_I$, where H_0 is the Hamiltonian of a free scalar field $\phi(x)$ of mass m in two-dimensional space-time and $H_I = \int : \phi^4(x) : g(x)\, dx$, where $\lambda > 0$ and $g(x)$ is a smooth function equal to 1 for small x and 0 for large x. The symbol : : denotes Wick ordering. Glimm and Jaffe [2] have shown that H and H_I are essentially self-adjoint on appropriate domains. We cannot apply the method directly to H_I because H_I is not bounded from below. Although Glimm [8] has shown, using a functional integration technique due to Nelson [9], that the operators $H = H_0 + H_I$ and $N + H_I$ are semibounded, we will see that vectors in $C^\infty(H)$ do not have a simple form. By introducing momentum cutoffs in either H_0 or H_I, we obtain approximate operators H' and H'' for which these problems do not arise.

We represent the operators in the Fock space $\mathscr{F} = \mathscr{F}_0 \oplus \mathscr{F}_1 \oplus \cdots$, where $\mathscr{F}_n$ is the "n-particle subspace" of functions $\phi_n(k_1, \ldots, k_n)$ which are symmetric under interchange of variables and square-integrable: $\int \cdots \int dk_1 \cdots dk_n \mid \phi_n(k_1, \ldots, k_n)|^2 = \| \phi_n \|^2 < \infty$. $\mathscr{F}_0$ is the complex numbers. Elements of $\mathscr{F}$ are written

$$\Phi = [\phi_0, \phi_1(k), \ldots, \phi_n(k_1, \ldots, k_n), \ldots]$$

with the norm $\| \Phi \|^2 = \sum_{n=0}^{\infty} \| \phi_n \|^2$. We will write $\Phi^{(n)} = \phi_n \in \mathscr{F}_n$ and $\Phi_n = [0,..., 0, \phi_n(k_1 ,..., k_n), 0,...] \in \mathscr{F}$.

In Fock space, we have

$$H_0 = \int a^+(k)\, a(k)\, \mu(k)\, dk,$$

$$H_I = \int \cdots \int dk_1 \cdots dk_4\, W(k_1 ,..., k_4) :\prod_{i=1}^{4} (a^+(k_i) + a(-k_i)): .$$

Here $\mu(k) = \sqrt{k^2 + m^2}$ and $W(k_1 ,..., k_4)$ is a symmetric function with $\int | W(k_1 ,..., k_4)|^2\, dk_1 \dots dk_4 = \| W \|^2 < \infty$. We also introduce the number operator $N = \int a^+(k)\, a(k)\, dk$. The operator $a(k)$ is defined by

$$(a(k)\Phi)^{(n-1)}\, (k_1 ,..., k_{n-1}) = n^{1/2} \Phi^{(n)}(k, k_1 ,..., k_{n-1}).$$

$a^+(k)$ is formally adjoint to $a(k)$.

Using the definitions of $a(k)$, Φ_n, and $\Phi^{(n)}$, one may calculate

$$\begin{aligned}(N\Phi)^{(n)} &= n\Phi^{(n)},\\ (H_0\Phi)^{(n)}\, (k_1 ,..., k_n) &= [\mu(k_1) + \cdots + \mu(k_n)]\, \Phi^{(n)}(k_1 ,..., k_n),\end{aligned} \tag{8}$$

and, for example,

$$\begin{aligned}&(H_I\Phi_n)^{(n+4)}\, (k_1 ,..., k_{n+4})\\ &\quad = S[(n+4)(n+3)(n+2)(n+1)]^{1/2}\, W(k_1 ,..., k_4)\, \Phi_n^{(n)}(k_5 ,..., k_{n+4}).\end{aligned} \tag{9}$$

Here S is a "symmetrizing operator" with $\| S \| = 1$.

The space C of vectors $\Phi \in \mathscr{F}$ of the form $\Phi = [\phi_0 ,..., \phi_n , 0, 0,...]$ forms a common, invariant, dense domain for the operators N and H_I. However, the function $W(k_1 ,..., k_4)$ does not decrease rapidly for large values of $| k_i |$ and consequently the product $\mu(k)\, W(k_1 ,..., k_4)$ is not square integrable. In view of (8) and (9) this means we cannot define the operators $H_0 H_I$ or $(H_0 + H_I)^n$ on C.

To construct our first approximate Hamiltonian we replace the function $\mu(k)$ in H_0 by a function which approaches a constant for large k. Essentially, we take $H' = N + H_I$. As before we may write $H' = \sum_k W_k$ with $[N, W_k] = kW_k$ ($| k | = 4, 2, 0$). Applying the Schwarz inequality to Eqs. (8) and (9) it is not hard to establish that the inequality $\| W_k \Phi \| \leqslant c \|(N + 4)^2\, \Phi \|$ holds for $| k | = 4, 2, 0$, $\Phi \in C$ and $c = (\frac{1}{8} + 6 \| W \|)$. Thus the operators N and W_k satisfy the conditions of Corollary 1 with $W = H'$, so $N + H_I$ is essentially self-adjoint on C.

We now define $C_L = \{\Phi \mid \Phi \in C, \Phi^{(n)}(k_1, ..., k_n) = 0 \text{ if } |k_i| > L, i = 1, ..., n; n = 1, 2, ...\}$. If $\Phi \in C_L$, we have, from Eq. (8),

$$\| H_0 \Phi \| \leqslant \mu(L) \| N\Phi \| \leqslant \tfrac{1}{8}\mu(L) \|(N+4)^2 \Phi\|. \tag{10}$$

We introduce the cut-off operator:

$$H_{Ip} = \iiiint_{-p}^{p} W(k_1, ..., k_4) : \prod_{i=1}^{4} (a^+(k_i) + a(-k_i)) : dk_1 \cdots dk_4 .$$

(This corresponds to using a "momentum cut-off field" $\phi_p(x) = (4\pi)^{-1/2} \int_{-p}^{p} [a^+(k) + a(-k)] \, e^{-ikx} \mu(k)^{-1/2} \, dk$ in place of $\phi(x)$ in H_{Ip}.

If $\Phi \in C_L$, it follows from Eq. (7) that $H_0 \Phi \in C_L$ and $N\Phi \in C_L$. If we consider the equation corresponding to (9) for H_{Ip} instead of H_I, we see that $H_{Ip}\Phi$ is also in C_L provided that $L > p$. Thus C_L is a common invariant domain for H_0, N, and H_{Ip}. If we take $H'' = H_0 + H_{Ip} = \sum_k W_k$ with $[N, W_k] = kW_k$, then using (10) we obtain the inequalities $\| W_k \Phi \| \leqslant c_L \|(N+4)^2 \Phi\|$ for $|k| = 4, 2, 0$, $\Phi \in C_L$, with $c_L = [\mu(L)/8 + 6 \| W \|]$. Glimm's proof that $H_0 + H_I$ is semibounded also applies to $H_0 + H_{Ip}$. It is not hard to see that $\bigcup_{L \to \infty} \{C_L\}$ is dense in $\mathscr{F}$. Thus by Corollary 1 with $W = H''$ we have that H'' is essentially self-adjoint on $\bigcup_{L \to \infty} \{C_L\}$.

IV. Solution of Linear Equations

Using the polarization identity

$$\begin{aligned} \langle Tu \mid v \rangle = \tfrac{1}{4}[&\langle T(u+v) \mid (u+v) \rangle - \langle T(u-v) \mid (u-v) \rangle \\ &+ i\langle T(u+iv) \mid (u+iv) \rangle - i\langle T(u-iv) \mid (u-iv) \rangle], \end{aligned}$$

we may reduce the solution of the linear equation $(1 - zA)\, h = \phi$, where A is self-adjoint and $\operatorname{Im} z \neq 0$, to the evaluation of the diagonal matrix elements $\langle f \mid (1 - zA)^{-1} f \rangle$ of the bounded operator $(1 - zA)^{-1}$ for all f in some dense set of vectors. Theorems 1 and 2 tell us that A has a dense set $\mathscr{U}$ of vectors of uniqueness and give us simple characterizations of certain dense subsets of $\mathscr{U}$. This is useful because it can be shown that the Padé approximant to the formal power series $\sum_{n=0}^{\infty} z^n \langle f \mid A^n f \rangle$ converges to $\langle f \mid (1 - zA)^{-1} f \rangle$ provided that f is a vector of uniqueness for A. We briefly indicate this connection.

Let f be a vector of uniqueness for A. Then, in the notation of Section II, $B = (A \mid \mathscr{L}_f)^-$ is self-adjoint in $\mathscr{H}_f$, and it is not hard to see that $(1 - zA)^{-1} g = (1 - zB)^{-1} g$ for all $g \in \mathscr{H}_f$. It is easy to check that the inverse of $(\lambda - B_n)$ in $\mathscr{H}_n$ is given by a polynomial of degree $n - 1$ in B_n :

$$R_{n-1}(B_n) = \frac{1}{p_n(\lambda)} \frac{p_n(\lambda) - p_n(B_n)}{\lambda - B_n}.$$

This operator is called the Method of Moments approximant to $(\lambda - B)^{-1}$ [10]. $R_{n-1}(B_n)$ converges strongly to $(\lambda - B)^{-1}$ on the range of $(\lambda - A \mid \mathscr{L}_f)$. To see this let $g = \sum_{k=0}^{M} a_k e_k \in \mathscr{L}_f$ and note that $Bg = B_n g$ if $n > M + 1$. Then

$$\begin{aligned}
&[R_{n-1}(B_n) - (\lambda - B)^{-1}](\lambda - A \mid \mathscr{L}_f) g \\
&\quad = R_{n-1}(B_n)(\lambda - B) g - g \\
&\quad = R_{n-1}(B_n)(\lambda - B_n) g - g \qquad \text{if} \quad n > M + 1 \\
&\quad = 0.
\end{aligned}$$

Since $\mathscr{R}(\lambda - A \mid \mathscr{L}_f)$ is dense in $\mathscr{H}_f$ (because $A \mid \mathscr{L}_f$ is essentially self-adjoint) and $(\lambda - B)^{-1}$ is bounded, this extends to all of $\mathscr{H}_f$. In particular,

$$\begin{aligned}
\langle f \mid R_{n-1}(B_n) f \rangle &= \langle f \mid R_{n-1}(B) f \rangle \\
&= \frac{1}{p_n(\lambda)} \left\langle f \,\middle|\, \frac{p_n(\lambda) - p_n(B)}{\lambda - B} f \right\rangle \\
&= \frac{q_n(\lambda)}{p_n(\lambda)} \xrightarrow[n \to \infty]{} \langle f \mid (\lambda - B)^{-1} f \rangle,
\end{aligned}$$

where $q_n(\lambda)$ is a polynomial of degree $n - 1$ in λ.

If we put $z = 1/\lambda$ and $N_{n-1}(z) = w_n z^{n-1} q_n(1/z)$ and $D_n(z) = w_n z^n p_n(1/z)$, then $N_{n-1}(z)/D_n(z)$ is the $[n, n - 1]$ Padé approximant to the formal power series expansion for $\langle f \mid (1 - zA)^{-1} f \rangle$. (This means that N_{n-1}/D_n is the unique ratio of a polynomial N_{n-1} of degree $n - 1$ to a polynomial D_n of degree n (with $D_n(0) = 1$) whose power series expansion agrees with the formal power series for the first $2n$ terms.)

Acknowledgments

We would like to thank Professor A. Grossmann for some valuable discussions, Professor E. Prugovečki for a critical reading of the manuscript, Professor E. Nelson for bringing the work of A. E. Nussbaum to our attention, and Professor B. Simon for pointing out an error.

Note added in proof. After completing this work it was brought to the authors attention that Theorem 2 has also been proved by A. E. Nussbaum in *Studia Mathematica*, **33**, 1969, 305.

REFERENCES

1. E. NELSON, *Ann. of Math.* **70** (1959), 572–615.
2. J. GLIMM AND A. JAFFE, *Phys. Rev.* **176** (1968), 1945–1951.
3. J. GLIMM AND A. JAFFE, *Commun. Pure Appl. Math.* **22** (1969), 401–415.
4. A. E. NUSSBAUM, *Arkiv für Matematik* **6** (1965), 179–191.
5. M. H. STONE, "Linear Transformations in Hilbert Space," American Mathematical Society, Providence, R. I., 1964.
6. G. POLYA, "Patterns of Plausible Inference," Vol. II, Princeton University Press, Princeton, N. J., 1954.
7. N. I. AKHIEZER, "The Classical Moment Problem," Oliver and Boyd, Edinburgh and London, 1965.
8. J. GLIMM, *Commun. Math. Phys.* **8** (1968), 12–25.
9. E. NELSON, *In* "Mathematical Theory of Elementary Particles," (R. Goodman and I. Segal, Eds.), pp. 69–73, M.I.T. Press, Cambridge, Mass., London, 1965.
10. YA. V. VOROBYEV, "Method of Moments in Applied Mathematics," Gordon and Breach, New York, 1965.